COMPLETELY CONTINUOUS ELEMENTS OF A NORMED RING

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ON NORMED RINGS

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THESIS

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1. The contents of this paper arise almost entirely from a study of a particular class of elements of a normed ring, namely those which are completely continuous considered as operators on the underlying Banach space. Though the concept of a completely continuous element is thus primarily topological and though we base much of our work on theorems on linear operators due to F. Riesz and Banach, our methods are predominantly algebraic in that once this notion and those theorems are accepted and restated in the language of ideal theory, we proceed as though we were doing simple algebra. Complete continuity of an element not in the radical turns out very nicely to take the place of the classical descending chain condition in enabling us to construct an idempotent in an extremely direct fashion. It is interesting to note that this result can also be deduced using what we may term almost completely analytical methods, in contradistinction with ours, which we have called algebraic, from a theorem of Lorch which he proves with the Cauchy integral as his principal tool.

It is always a temptation, given a theorem on Banach spaces, to see whether something analogous can be proved replacing the norm by the weak topology. Only a small measure of success was attained in making this attempt in the present paper; it seems likely that one of the theorems referred to above, and which was essential to our proof, is definitely not true in the weak case though a less stringent form of it generalizes to the weak topology quite directly, and that consequently an entirely different approach will be needed to obtain the result if it holds at all.

Section 2 is preliminary in nature. We summarize our notation, list some known theorems and develop a few general ideas for later use. Section 3 deals with completely continuous elements, and §4 develops the corresponding theorems, as far as possible, for the weak case.

2. Our notation is standard and we summarize it here only for the sake of completeness. If A and B are sets, $A \cup B$ and $A \cap B$ are their union and intersection respectively. $A \subseteq B$ means that A is a subset of B, and $A \subset B$ that A is a proper subset of B. Convergence is denoted by $\rightarrow$. The norm of x is written $\| x \|$, the set of elements satisfying a condition is $\{x \mid \text{condition}\}$, and in this notation the unit sphere is $\{x \mid \| x \| \leq 1\}$ and will be denoted by S. Topological

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terms, unless qualified, refer to the norm topology. The ring operations being denoted by $+$ and $\cdot$, $A + B = \{z \mid z = a + b, a \in A, b \in B\}$, $AB = \{z \mid z = ab, a \in A, b \in B\}$. Whenever we refer to the dimension of a vector space it will be in the sense of the cardinal of a Hamel basis for the space. The symbols $C(0, 1)$, $M(0, 1)$, and m refer to the complex Banach spaces of uniformly continuous functions, bounded functions, both on the unit interval, and of bounded sequences of complex numbers respectively, with the usual definitions of the vector operations and norm [1; 10–11]. The following theorems will be assumed known.

THEOREM 2.1. *A Banach space is finite-dimensional if and only if the unit sphere S is compact.*

THEOREM 2.2. *A Banach space is reflexive if and only if the unit sphere S is weakly compact.*

In this last theorem, weak sequential compactness is sufficient, due to a recent result of Eberlein [2].

We deal with a normed ring R in the sense of Gelfand [3], that is, with a commutative Banach algebra over the field of complex numbers with unit-element e (the zero of the additive group will be denoted by θ). The correspondence $t \leftrightarrow T$ defined by $tx = T(x)$ for all $x \in R$ is a topological isomorphism of R onto a subring of the ring L of linear bounded operators in R considered as a Banach space. The continuity of ring multiplication has been shown by Gelfand [3] to imply and be implied by the existence of an equivalent norm satisfying $\| xy \| \leq \| x \| \| y \|$ for all $x, y \in R$. This norm is just that induced in R when considered as a subring of L under the above isomorphism: it follows that when the norm is so chosen then this mapping is not only a topological isomorphism but an isometry in the sense of Banach, and R may henceforth be considered a closed subring of the ring L. As a result, we may, without loss of generality, make use of the language and results of operator theory and shall do so throughout whenever convenient. In this connection we shall say that an operator T of L is "in R" whenever T is the correspondent under the natural imbedding (which Lorch has called the "regular representation of R") of an element $t \in R$, that is, when R contains an element t such that $tx = T(x)$ for all $x \in R$. Gelfand [3] has given a useful criterion for an operator's being "in R", as follows.

THEOREM 2.3. *T is "in R" if and only if $T(xy) = T(x)y$ for all $x, y \in R$.*

The question naturally arises whether an element of R which has an inverse in L has this inverse "in R". The above criterion permits us to answer this question in the affirmative: if T is "in R" and T^{-1} exists, then $xy = [T(T^{-1}x)]y = T[(T^{-1}x)y]$ since T is "in R". Applying T^{-1} to each side we get $T^{-1}(xy) = T^{-1}(x) \cdot y$, whence T is "in R", by Theorem 2.3. We call an element of R regular or a unit if it has a multiplicative inverse in R. Hence we have

THEOREM 2.4. *$t \in R$ is a unit in L if and only if t is a unit in R.*

The notion of quasi-regularity, introduced into the study of topological rings by Kaplansky [4], serves to make some of our discussion much briefer than it would otherwise be. Its usefulness in Kaplansky's paper derives largely from the fact that it aids in getting around the difficulties due to the absence of a unit-element. In the present case we have an identity but quasi-regularity supplies us with a rather neat way of getting from elements of the radical to units. How this happens is evident from the relevant definitions and theorems which we summarize here following [4].

DEFINITION 2.1. $x \in R$ is quasi-regular (q.r.) if there exists an element $y \in R$ such that $x + y + xy = \theta$. y is called a quasi-inverse of x.

THEOREM 2.5. *x is* q.r. *if and only if $x + e$ is regular.*

This is evident from the fact that if $x + y + xy = \theta$ then $(x + e)(y + e) = e$.

The Gelfand radical is the same as the radical defined as the union of all the q.r. ideals of R, an ideal being q.r. if all its elements are. (See [4].) Hence an element of a normed ring is in the radical if and only if the principal ideal generated by it consists entirely of q.r. elements. Putting this another way gives

THEOREM 2.6. *$x \in$ Rad R if and only if $rx + e$ is regular for all $r \in R$.*

It follows, in particular, that $x \in \text{Rad } R$ implies $(x - \lambda e)^{-1}$ exists for all scalars $\lambda \neq 0$, in other words, that the spectrum of an element in the radical consists of the point $\lambda = 0$ alone.

Given an ideal I we denote by A_I the annihilator ideal of I, that is, the set of all elements of R which annihilate every element of I. If I is the principal ideal generated by the element x we denote it by $\{x\}$ and its annihilator by A_x. For later reference we note the following: I^n denotes the smallest ideal containing all n-th powers of elements in I, $I \cdot I \cdot \cdots \cdot I$ (to n factors) the smallest ideal containing all products of n factors, each in I; and $I \supseteq I \cdot I \supseteq I^2$. For principal ideals this simplifies to $\{x\}^n = \{x^n\} = \{x\} \cdot \cdots \cdot \{x\}$. Finally we observe that principal ideals satisfy $\{x\} \cap \{e - x\} = \{x - x^2\}$ and $A_x \cap A_{e-x} = \theta$, for $r \in A_{e-x}$, and $r \neq \theta$ imply $r - xr = \theta$ and hence that $xr \neq 0$.

3. A completely continuous operator on a Banach space is one which takes bounded sets into conditionally compact sets, that is, into sets having compact closure. An element of R is completely continuous (c.c.) if it is completely continuous considered as an operator on R, that is,

DEFINITION 3.1. $r \in R$ is c.c. if rB has compact closure for every bounded set $B \subset R$.

This is evidently equivalent to the condition that rS be conditionally compact where S is the unit sphere. Even though S is closed, rS will not, in general, be compact, for multiplication by ring elements does not necessarily take closed sets into closed sets except in the obvious case when the multiplier is regular, and in some other, less obvious, cases which we shall meet later. Completely continuous elements, moreover, cannot be units in an infinite-dimensional ring.

THEOREM 3.1. *A unit is* c.c. *if and only if R is finite-dimensional.*

For the proof we note that if R is finite-dimensional then S is compact (Theorem 2.1) and hence every bounded set is conditionally compact. Since multiplication preserves boundedness of subsets, every element is consequently c.c. and in particular, every unit is. Conversely, if a unit is c.c. it will take the set of products of S by its inverse into S, S must hence be compact, and by Theorem 2.1, R is finite-dimensional. That the existence of the unit element was not used superfluously in the above proof is seen by considering the ring obtained from an arbitrary infinite-dimensional Banach space by defining the product of any two elements to be zero. This is clearly a normed ring without unit which is infinite-dimensional and in which every element is c.c. In the sequel we deal only with infinite-dimensional rings.

Since the difference of c.c. elements and the product of a c.c. element by an arbitrary element of the ring are clearly c.c., the set I of all c.c. elements of the normed ring is an ideal. By a theorem of Banach [1; 96, Theorem 2] the limit of a sequence of c.c. operators is a c.c. operator, and since R is a closed subring of L, the limit of a sequence of c.c. elements is again a c.c. element. Hence, we have

THEOREM 3.2. *The set of all* c.c. *elements of a normed ring is a closed ideal I.*

The range of a ring element considered as a linear operator is simply the principal ideal generated by it. $\{r\} \cap S$ is clearly the unit sphere of $\{r\}$ considered as a normed ring (but since $\{r\}$ may not be closed, this ring may lack completeness) and if this is conditionally compact (in R) then since $rS \subseteq \{r\} \cap S$, r is c.c. But this is asking a lot; if the sphere of $\{r\}$ is compact, $\{r\}$ is finite-dimensional, that is to say, r considered as an operator has a finite-dimensional range. Hence we make

DEFINITION 3.2. r is finite-dimensional if $\{r\}$ is finite-dimensional.

Finite-dimensional elements are c.c. since the unit sphere in their range is compact, as noted above, but the converse is not true. The element $x = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$ which is not a finite-dimensional element of the space m of bounded sequences of numbers (multiplication is defined coordinatewise) may be thought of as the limit of the sequence $[x_n]$ where $x_n = (1, \frac{1}{2}, \cdots, 1/n, 0, 0, \cdots)$ are all finite-dimensional. The x_n, by what has been said, are all c.c. and so is their limit x (Theorem 3.2). Hence though the set F of all finite-dimensional elements is again clearly an ideal and is contained in I, it is not, in general, closed. A well-known conjecture asserts that $\overline{F} = I$.

A theorem of Banach [1; 152, Theorem 12] translated into our terminology gives

THEOREM 3.3. *If $r \in R$ is* c.c. *then A_{e-r} is finite-dimensional.*

It follows immediately that if x^{n-1} is c.c. and $x = x^n$ then x is finite-dimensional, and from this we get

COROLLARY 3.1. *An idempotent which is* c.c. *is also finite-dimensional.*

Furthermore, finite-dimensional elements are zero-divisors. That this is so is intuitively evident from the fact that such an element maps the whole space into a finite-dimensional subspace and hence must annihilate a great many elements. That this is indeed the case is seen as follows: if x is finite-dimensional then $\{x\}$ has a finite linear basis, that is, $\{x\}$ contains elements $x_1, x_2, \cdots, x_n$ such that for any $r \in R$ scalars $\lambda_1, \lambda_2, \cdots, \lambda_n$ may be found such that $rx = \sum_1^n \lambda_i x_i$. Since the x_i are in $\{x\}$ there exists a fixed set of elements $r_1, r_2, \cdots, r_n \in R$, such that $x_i = r_i x$, $i = 1, 2, \cdots, n$. Hence for any $r \in R$ we can write $rx = \sum_1^n \lambda_i r_i x$ or $x(r - \sum \lambda_i r_i) = \theta$. The vanishing of the bracket would consequently imply the existence of a finite linear basis for the ring R, which we supposed to be infinite-dimensional. Thus x must be a divisor of zero. Zero-divisors need not, however, be finite-dimensional, as shown by the following example: we make a normed ring out of the space $C(0, 1)$ of continuous functions on the closed unit interval by defining multiplication to be pointwise; the unity of this ring is then the function which is identically one. $C(0, 1)$ has dimension $\aleph_0$, and contains lots of zero-divisors, but no finite-dimensional elements. We note first that non-regular elements of this ring are functions having at least one zero, and that if a function in $C(0, 1)$ is not identically zero then it differs from zero on an open set and hence on some closed sub-interval (a, b) of an open interval in it. On this interval, however, the function is regular and generates there the space $C(a, b)$ on this interval. Since $C(a, b)$ contains as many elements as $C(0, 1)$ itself, no element of $C(0, 1)$ is finite-dimensional. As we shall see later, the space $C(0, 1)$ does not even contain c.c. elements. Any continuous function, however, which vanishes over some interval, is a zero-divisor. For later reference we note that $C(0, 1)$ contains no idempotents other than the functions which are identically one or zero. In the ring of bounded functions, there are, of course, many: every characteristic function is idempotent, multiplication being defined as in the space $C(0, 1)$. Furthermore, $C(0, 1)$, $M(0, 1)$, as well as any other space in which multiplication is of a pointwise nature can contain no proper nilpotents. This suggests the surmise that some connection exists between the presence of finite-dimensional and nilpotent elements in an infinite-dimensional ring. That finite-dimensional elements need not be nilpotent is easily seen in the space m considered above. Since in the spaces $C(0, 1)$, $M(0, 1)$, m, etc., there are no nilpotents at all, it is trivially true in these that nilpotency implies finite-dimensionality, but we have been unable to show this to be generally true.

A commutative ring R is said to be reducible if it contains a proper idempotent (we consider θ and e improper). The following is known to be equivalent to this property and we take it for our definition.

DEFINITION 3.3. R is reducible if R is the direct sum of subrings R_1 and R_2 where the R_i are principal ideals generated by orthogonal idempotents.

If r is any non-unit of R then it is always true that A_r is a subset of $\{e - r\}$, and reducibility follows when $A_r = \{e - r\}$, for then $r = r^2$ whence r is the required idempotent.

Since we shall be concerned in what follows with non-trivial ideals only, any elements with which we deal will be supposed to be non-units unless the contrary is explicitly stated. We note first that if $a \in R$ is any element, than $A_a + \{a\} = R$ implies the existence of an element $r \in R$ such that $e - ra \in A_a$, and conversely; for if $\{a\} \neq R$ and $A_a + \{a\} = R$ then there exists an element $r \in R$ such that $e = z + ra$ for some $z \in A_a$ where $ra \neq \theta$. Hence $(e - ra) \in A_a$, that is to say $a - ra^2 = \theta$ (or $ra - (ra)^2$). To prove the converse we need merely note that if $e - ra \in A_a$ where $r \notin A_a$, then $e - ra = z$ for some $z \in A_a$ and consequently $e \in A_a + \{a\}$, ra not being zero. When $r = e$ works, this reduces to the result of the preceding paragraph.

Making use of some remarks at the end of §2, we see that for any $q \in R$ we always have two corresponding chains of ideals:

$$R \supseteq \{q\} \supseteq \{q^2\} \supseteq \{q^3\} \supseteq \cdots \supseteq \{q^n\} \supseteq \cdots \supseteq 0,$$

(*)

$$0 \subseteq A_q \subseteq A_{q^2} \subseteq A_{q^3} \subseteq \cdots \subseteq A_{q^n} \subseteq \cdots \subseteq R.$$

Clearly $A_{q^n} \subset A_{q^{n+1}}$ implies that $\{q^{n+1}\} \subset \{q^n\}$ where the inclusion is proper in both cases. Actually we are able to make the somewhat stronger statement of

THEOREM 3.4. *If* (a) *n is the least integer for which $A_{q^n} = A_{q^{n+1}}$ and* (b) *$\{q^N\} = \{q^{N+1}\}$ for some least integer $N \geq n$, then $n = N$.*

Proof. $\{q^N\} = \{q^{N+1}\}$ implies the existence of an element r such that $q^N = rq^{N+1}$, $q^N(e - rq) = \theta$, whence $(e - rq) \in A_{q^N} = A_{q^{N+1}} = \cdots = A_{q^n} = A_{q^{n+1}}$. Hence $(e - rq) \in A_{q^n}$, that is $q^n = rq^{n+1}$ and $\{q^n\} \subseteq \{q^{n+1}\}$ whence $\{q^n\} = \{q^{n+1}\}$ since the opposite inclusion always holds. But since N was the *least* integer for which this was true, $N = n$.

The next two theorems, of fundamental importance in what follows, are due essentially to F. Riesz, who proved them for completely continuous operators on the Banach space of continuous functions [7] but realized that they would apply in far more general situations. In Banach's book he gives a proof of the first of these for c.c. operators on an arbitrary Banach space, which is somewhat more concise though not essentially different [1; 151, Theorem 11]. The proof of the second is in [7; 85]. Before stating them we remark that the first is essential to the proof of the second, and though we phrase them in ring terminology, the proofs go over to the present case almost unchanged.

THEOREM 3.5. *If $t = e - u$ where u is* c.c., *then $\{t\}$ is closed.*

THEOREM 3.6. *If $t = e - u$ where u is* c.c., *then there exists an integer $N \geq 0$ such that $\{t^N\} = \{t^{N+i}\}$ if $i > 0$ and $\{t^n\} \supset \{t^{n+1}\}$ if $n < N$.*

Before going on we need assurance that when equality sets in, as asserted by Theorem 3.6, in the chain of principal ideals of (*) after a finite number of steps, it is not because the chain has ended in the zero ideal. Our next theorem guarantees this.

THEOREM 3.7. *If u is* c.c. *then $t = e - u$ is not nilpotent.*

Proof. If the theorem is untrue, then an integer n exists such that $t^n = \theta$. But $t^n = (e - u)^n$ implies that $\theta = t^n = e - u^*$ where u^* is c.c. as seen from the binomial expansion of $(e - u)^n$, whence $e = u^*$ and so e is c.c., which, by Theorem 3.1, is impossible in an infinite-dimensional ring.

We study separately the cases when the chains of (*) do not and when they do get started. In the case when they do not ($N = 0$), the ideal $\{t\}$ is the unit ideal, t is a unit, and $(-u)$ is q.r. (Theorem 2.3). If ur is q.r. for every $r \in R$ then the principal ideal $\{u\}$ consists entirely of q.r. elements, whence u is in the radical. If this is not the case, then for some $r \in R$, $u' = ur$ is *not* q.r., $t' = e - u'$ is *not* a unit, and for the ideal $\{t'\}$ we have $N > 0$.

We can prove Theorem 3.7 in a somewhat more elegant fashion by essentially the same method with which we just treated the case $N = 0$. For if $(e - u)$ is nilpotent (or even a generalized nilpotent in the sense of Gelfand), then as shown by Kaplansky [4], $e - u$ is in the radical, whence $(-u)$, and thus u, is a unit, contrary to the assumption of infinite-dimensionality of R. The cases $N = 0$ and $(e - u)$ in the radical are thus similar: they are the only ones in which the chains (*) are trivial, the one time because $e - u$ is a unit, the other because u is.

We suppose now that u is c.c. and neither a unit nor in the radical. Then for some $ru = u'$ (which we shall call u) if not for u itself, equality occurs in the chain of principal ideals of (*) for some least integer $N > 0$.

Clearly the corresponding chain of annihilators must cease to increase for some $n \leq N$. Actually, by Theorem 3.4, we must have $n = N$. Under these conditions we can now assert

THEOREM 3.8. $R = \{t^N\} \oplus A_{t^N}$ ($\oplus$ *denotes a direct sum*).

Proof. Since $\{t^N\} = \{t^{2N}\}$ there exists an element $r \in R$ such that $t^N = rt^{2N}$; hence $rt^N = (rt^N)^2$, whence $rt^N = e_1$ is an idempotent. The element $e_2 = e - e_1$ is an orthogonal idempotent, and so $R = \{e_1\} \oplus \{e_2\}$. Since $e_1 = rt^N$, $\{e_1\} \subseteq \{t^N\}$. But since $t^N = rt^{2N} = e_1 t^N$, we also have $\{t^N\} \subseteq \{e_1\}$ whence $\{e_1\} = \{t^N\}$. Similarly, $\{e_2\} \subseteq A_{t^N}$ because $e_2 e_1 = \theta$ and since, as noted earlier, the opposite inclusion always holds, we have that $A_{t^N} = \{e_2\}$, whence the theorem follows. Combining the result of Theorem 3.8 with some of the remarks leading up to it we have

COROLLARY 3.2. *If R is irreducible every* c.c. *element is in the radical of R.*

As remarked in the introduction, this theorem can also be deduced from the theorem of Lorch [5; Theorem 2] that a ring is irreducible if and only if the spectrum of every element is a connected set. For, since the spectrum of a c.c. element is discrete, that of a c.c. element of an irreducible ring can consist of *at most* one point. But since c.c. elements of an infinite-dimensional ring are non-units, their spectra must contain *at least* the one point $\lambda = 0$, and hence, by another of Lorch's theorems [5; Theorem 6], such elements are in the radical.

It is of interest to note that no use was made, once Theorem 3.6 was obtained, of the fact that u was c.c. and $t = e - u$ in proving the reducibility theorem. Hence we have shown that if R contains a non-unit t satisfying (a) $t^n \neq \theta$ for all n, and (b) there exists an integer N such that $\{t^N\} = \{t^{N+1}\}$, where N is minimal with this property, then R is reducible, $R = A_{t^N} \oplus \{t^N\}$, $A_{t^N} = A_{t^{N+i}}$, $i \geq 0$, and $A_{t^n} \subset A_{t^{n+1}}$ for $n < N$. Hence, in particular, if R contains a non-unit which is not in the radical, and satisfies a descending chain condition, then R is reducible. The direct summands of a reducible ring are necessarily closed (no matter what Hausdorff topology in the ring), for each is the full inverse image under a continuous mapping of a single point.

If we apply Corollary 3.2 to the space $C(0, 1)$ we see that none of its elements are c.c., for, as we observed earlier, it is an irreducible ring and its radical is the zero ideal. This implies in particular that there are no principal ideals $\{t\}$, $t = e - u$ and u c.c. Such ideals would be closed (Theorem 3.5). But we can actually show more—that no principal ideals of the ring $C(0, 1)$ are closed. To do so we state first a generalization by Bourgin of an unpublished result of Kaplansky.

THEOREM 3.9. *If R is a ring of real- or complex-valued functions on a discrete space, which are bounded, and in which multiplication is pointwise, with* $\| x \| =$ l.u.b. $| x(t) |$, *then closed ideals J satisfy $J = J^2$.*

Bourgin's proof is so simple that we reproduce it: it suffices to consider the complex case and to show that $f^2 \in J$ implies $f \in J$. We write $f(t) = p(t)e^{i\varphi(t)}$. Then $f^2 \in J$ implies $f^3 \in J$ and also, for any $x \in R$, $f^3/x \in J$ provided $| x(t) | \geq \epsilon > 0$ for all t. Hence $Q_\epsilon = [P(t)]^3 e^{3i\varphi(t)}/\{[P(t)]^2 + \epsilon\}e^{2i\varphi(t)}$ is in J for all $\epsilon > 0$ and $\| Q_\epsilon - f \| \to 0$ as $\epsilon \to 0$. Since J is closed, it follows that $f \in J$. Since $J \supseteq J \cdot J \supseteq J^2$ always holds, we also have $J = J \cdot J$. It is this result which suggests the following theorem.

THEOREM 3.10. *If R is irreducible and $J = J^2$ for closed ideals J, then no principal ideal of R is closed.*

For the proof observe that assuming the contrary implies the existence of an idempotent as above, which is impossible in an irreducible ring.

It follows that in a ring satisfying the conditions of Theorem 3.9 which is irreducible, principal ideals cannot be closed. The space $C(0, 1)$ is an example of such a ring; that the hypothesis of irreducibility is necessary is seen by considering the space $M(0, 1)$; since it is reducible, it is the direct sum of the principal ideals generated by orthogonal idempotents and, as remarked earlier, these are necessarily *closed* principal ideals.

4. We say that an element of a normed ring is weakly completely continuous (w.c.c.) if rB is weakly conditionally sequentially compact for every set B which is bounded (in the norm topology). For c.c. elements the fact that we impose only *conditional* compactness was not significant, for since R is (norm-) complete

the (norm-) closure of a conditionally compact set will be compact. The weak closure of a weakly sequentially conditionally compact set, however, will, in general, be compact only if the space is weakly sequentially complete. But the restriction to *sequential* compactness is not very serious since Eberlein [2] has shown that sequential compactness and bicompactness of weakly closed subsets are equivalent.

Letting $J = \{r \mid r \in R, r \text{ w.c.c.}\}$ we can prove

THEOREM 4.1. *J is a closed ideal.*

Proof. That J is an ideal is seen by the same argument as was used in showing I an ideal. To prove closure we use a slight modification of Banach's proof of the closure of I. We observe first that since an ideal is a subspace and hence convex, weak closure is equivalent with norm closure and we prove the latter. Let (t_n) be a sequence of elements of J which converges strongly to an element $t \in R$. If (x^n) is a sequence of elements of norm one then it contains a subsequence (x^{n_1}) such that $(t_1 x^{n_1})$ is a Cauchy sequence (in the weak topology) and a subsequence (x^{n_k}) of $(x^{n_{k-1}})$ such that $(t_l x^{n_k})$ is a (weak) Cauchy sequence for all $l \leq k$. Hence by the diagonal principle there exists a subsequence (x^{*m}) of all these for which $(t_n x^{*m})$ is a weak Cauchy sequence for all n. That is, for any functional f in (the conjugate space) R^* and any $\epsilon > 0$ there exists N_n^ϵ such that $| f[t_n(x^{*m} - x^{*m'})] | < \epsilon/3$ for $m, m' > N_n^\epsilon$. Now consider

$$
\begin{aligned}
A &\equiv | f[t(x^{*m} - x^{*m'})] | \\
&\leq | f[tx^{*m} - t_n x^{*m}] | + | f[t_n(x^{*m} - x^{*m'})] | + | f[t_n x^{*m'} - tx^{*m'}] | \\
&\leq \| f \| \, \| t - t_n \| \, (\| x^{*m} \| + \| x^{*m'} \|) + | f[t_n(x^m - x^{m'})] | \\
&= 2 \| f \| \, \| t - t_n \| + | f[t_n(x^{*m} - x^{*m'})] |
\end{aligned}
$$

since $\| x^{*m} \| = 1$ for all m. Since $t_n \to t$, there exists an integer N' depending on ϵ such that $\| t - t_n \| < \epsilon/3 \| f \|$ for $n > N'$. Hence choosing $n > N'$ and m, $m' > N_n^\epsilon$, we have that $A \leq \frac{2}{3} \| f \| \epsilon \| f \|^{-1} + \frac{1}{3}\epsilon = \epsilon$ and so (tx^{*m}) is a weak Cauchy sequence showing t to be w.c.c.

It is well known that in a Banach space properties which in the norm topology are equivalent with finite-dimensionality imply reflexivity when they are asserted in the weak topology. So, for example, norm compactness is equivalent with finite-dimensionality, weak compactness with reflexivity. We showed above that finite-dimensionality of rings with unit is equivalent to every element's being c.c. Hence the following theorem is not surprising.

THEOREM 4.2. *$J = R$ if and only if R is reflexive.*

Proof. If $J = R$ then the unit element is w.c.c., S is weakly compact, and R reflexive. Conversely, if R is reflexive, the sphere is compact (in the weak topology (see Theorem 2.3)), the unit element is w.c.c. and $J = R$.

Our proof makes use of the existence of the unit element in an essential fashion, as the following example shows: let R be any non-reflexive Banach space and define multiplication by making the product of any two elements equal to zero. Then R becomes a normed ring without unity in which clearly every element is w.c.c. (and in fact, also c.c.).

We should like to prove analogues to Theorems 3.5 and 3.6 for w.c.c. elements. The first of these, for which we referred to Banach's proof, is demonstrated by first showing that if u is c.c. then $t = e - u$ takes closed bounded sets into closed sets. That much generalizes directly, provided we are assured continuity of multiplication in one factor of the ring product in the weak sequential topology, and with the additional hypothesis of weak sequential completeness of R. We state and prove first a theorem which gives us the continuity needed. (The author is unable to prove the following conjecture: *Multiplication is continuous simultaneously in both factors of the product in the weak sequential topology if the radical of R is the zero ideal.*)

THEOREM 4.3. *In the weak sequential topology multiplication is continuous in each factor of the ring product.*

Proof. Since R is commutative we need merely show that $f(a_n) \to f(a)$ for all $f \in R^*$ implies $f(a_n b) \to f(ab)$ for all $b \in R$ and all f. Now to every f there corresponds for a given ring element b a linear functional f_b defined by $f_b(x) = f(bx)$. If we think of b as an operator B on R then f_b is the transform of f under the adjoint operator B^* of B, and so f_b is seen to be a linear functional. By hypothesis, then, $f_b(a_n) \to f_b(a)$ for every f_b, that is, for any b and f. Hence by definition of f_b we have $f(ba_n) \to f(ba)$ for all f.

Using Theorem 4.3 we can now prove

THEOREM 4.4. *If R is weakly sequentially complete, if u is* c.c., *$t = e - u$, and B is bounded and weakly sequentially closed, then tB is weakly sequentially closed.*

Proof. Suppose (x^n) is a sequence in B such that $f(tx^n) \to f(y)$ for all $f \in R$. We must show $y \in tB$. Since B is bounded, (x^n) contains a subsequence (x^{n_i}) such that $f(ux^{n_i}) \to f(x)$ for some $x \in R$ and all $f \in R^*$ since u is w.c.c. and R weakly sequentially complete. Hence $f(tx^{n_i}) + f(ux^{n_i}) \to f(y) + f(x)$ or $f[(t + u)x^{n_i}] \to f(y + x)$, f being additive. But since $t = e - u$ we have $f(x^{n_i}) \to f(y + x)$ whence $(y + x) \in B$ since B is weakly sequentially closed. With the continuity guaranteed by Theorem 4.3 we get $f(tx^{n_i}) \to f[t(x + y)]$ whence $y = t(x + y)$ the set of all f being total. Hence $y \in tB$ since $(x + y) \in B$.

In proving this theorem we have shown incidentally that if (tx^n) converges weakly then so does some subsequence (x^{n_i}) of the x^n. This seems to indicate that if B is not bounded we shall not be able to prove closure of tB but we have been unable to show this to be the case. Furthermore, it is clear that the foregoing proof applies somewhat more generally to any linear topological space which is a ring such that multiplication is continuous in each factor separately and which is complete with respect to a topology described in terms of convergence of se-

quences, boundedness of subsets B being interpreted to mean that for every open set U a scalar α can be found such that $B \subset \alpha U$.

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